Example of using Stokes Theorem in a boundary surface of a solid (called divergence thm in this case)

$$\iint_S V\cdot N\,dA = \iiint_R \underbrace{(\operatorname{div} V)}_{\nabla\cdot V}\, \underbrace{dV}_{dx\,dy\,dz}$$

$$\iint_{S=\partial R} \underbrace{\omega}_{\text{2-form}} = \iiint_R \underbrace{d\omega}_{\text{3-form.}}$$

3.22 $\displaystyle\iint_S B\cdot N\,dA$, S = boundary of

R = region between $z = x+2y$ & $z = x^2+y^2$

N is the outward unit normal.

$B = (z+3xy, -1, y)$

<u>Question</u>: Set up using the divergence thm.

$(V_1, V_2, V_3) \Leftrightarrow V_1\,dy\wedge dz + V_2\,dz\wedge dx + V_3\,dx\wedge dy$

$$\iint_S B\cdot N\,dA = \iint_S (z+3xy)\,dy\wedge dz + (-1)\,dz\wedge dx + y\,dx\wedge dy$$

Use divergence thm

$dx \wedge dy \wedge dz$

$$= \iiint_R \underbrace{\operatorname{div}(z + 3xy, -1, y)}_{\partial_x(z+3xy) + \partial_y(-1) + \partial_z(y) = 3y.} \, dx\,dy\,dz$$

$$= \iiint_R 3y \; dx\,dy\,dz$$

What is R?

Between $z = x + 2y$ and $z = x^2 + y^2$.

Where do they intersect in terms of x, y coordinates?

$$x + 2y = x^2 + y^2$$

$$0 = x^2 - x \quad + y^2 - 2y$$

$$+\tfrac{5}{4} \qquad +\tfrac{1}{4} \qquad +1$$

$$x^2 - ax + \frac{a^2}{4}$$

$$\left(\frac{a}{2}\right)^2 \quad = \left(x - \frac{a}{2}\right)^2$$

$$\frac{5}{4} = x^2 - x + \frac{1}{4} \quad + \quad y^2 - 2y + 1$$

$$\frac{5}{4} = \left(x - \frac{1}{2}\right)^2 + (y - 1)^2$$

Circle of radius $\frac{\sqrt{5}}{2}$ centered at $\left(\frac{1}{2}, 1\right)$

$$\frac{5}{4} - \left(x-\frac{1}{2}\right)^2 = (y-1)^2$$

$$\pm\sqrt{\frac{5}{4} - \left(x-\frac{1}{2}\right)^2} = y-1$$

$$y = 1 \pm \sqrt{\frac{5}{4} - \left(x-\frac{1}{2}\right)^2} \quad \leftarrow \text{2 y bounds.}$$

$$\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}$$

$$\text{Flux} = \boxed{\int_{x=\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \int_{y=1-\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}}^{1+\sqrt{\frac{5}{4}+\left(x-\frac{1}{2}\right)^2}} \left(\int_{z=x^2+y^2}^{x+2y} 3y \, dz \right) dy \, dx}$$

Integrating a function over a surface

$$\int_S f \, dA \approx \sum f(x,y) \, dA$$

$z = f(x,y)$ — how do we calculate dA?

surface point $\left(x, y, \overset{z}{f(x,y)}\right)$

$\frac{\partial}{\partial x}(\qquad)\,dx$

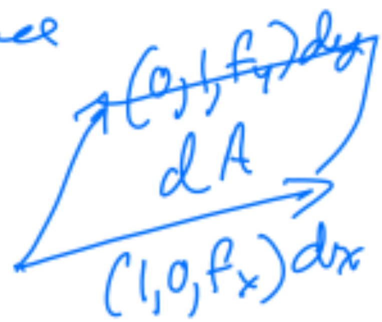
$(1, 0, f_x)\,dx$

↑ vector Δx on surface

Same with dy — vector on surface is $\frac{\partial}{\partial y}(\ )\,dy$

$(0, 1, f_y)\,dy$

on surface



$$dA = \|(1,0,f_x)\,dx \times (0,1,f_y)\,dy\|$$
$$= \|(1,0,f_x) \times (0,1,f_y)\|\,dx\,dy$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = i(-f_x) - j(f_y) + k(1)$$

$$= (-f_x, -f_y, 1)$$

$$\|(-f_x, -f_y, 1)\| = \sqrt{f_x^2 + f_y^2 + 1}$$

$$\boxed{dA = \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy}$$

To integrate a function $g(x,y,z)$ over the surface $z = f(x,y)$,

We compute

$$\iint\limits_{\substack{R \\ \text{xy plane}}} g\big(x, y, \overset{z}{f(x,y)}\big)\sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$$

Example 3.29

(a) Calculate $Q = \iint_M \overset{\text{fcn.}}{(y^2-3x)}\, dA$, where

M is the surface $\{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, z = \underbrace{x^2+2y^2}_{f(x,y).}\}$

$$dA = \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy$$

$$= \sqrt{(2x)^2 + (4y)^2 + 1}\; dx\, dy = \sqrt{4x^2+16y^2+1}$$

$$Q = \int_{x=0}^{1}\int_{y=0}^{1} (y^2-3x)\sqrt{4x^2+16y^2+1}\; dy\, dx$$

≈ -3.006 (sagemath)

(b) Calculate $\iint_M (y^2-3x)\, dx \wedge dy$

$\int V \cdot N\, dA$, where $V = (0, 0, y^2-3x)$

$z = f(x,y)$
$= x^2+2y^2$

dx dy

$$= \int_{x=0}^{1}\int_{y=0}^{1} (y^2-3x)\, \cancel{dy\, dx}$$

$$= \int_0^1 \frac{y^3}{3} - 3xy \Big|_0^1 \, dx$$

$$= \int_0^1 \frac{1}{3} - 3x \, dx$$

$$= \frac{x}{3} - \frac{3x^2}{2} \Big|_0^1 = \frac{1}{3} - \frac{3}{2}$$

$$= \boxed{\frac{-7}{6}}$$

(c) $\iint_M (y^2 - 3x)\, dx \wedge dz$

(like $V \cdot N \, dA$, where $V = (0, -(y^2-3x), 0)$)
$dz \wedge dx$

$$z = x^2 + 2y^2$$

$$dz = 2x\,dx + 4y\,dy$$

$$dx \wedge dz = dx \wedge (2x\,dx + 4y\,dy)$$

$$= 4y \, dx \wedge dy$$

$$\iint_M (y^2 - 3x)(4y \, dx \wedge dy)$$

$$= \iint_M (4y^3 - 12xy)\, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (4y^3 - 12xy)\, dx\, dy$$

$$= \int_{y=0}^{1} \left(4y^3x - 6x^2y\right)\Big|_{x=0}^{1} \, dy$$

$$= \int_{y=0}^{1} \left(4y^3 - 6y\right) dy$$

$$y^4 - 3y^2 \Big|_0^1 = 1 - 3 = \boxed{-2}$$

14. Let W be the vector field $W(x, y, z) = (z, -z + 1, 2x)$.

(a) Is W conservative?

(b) ~~Show that $W = \nabla \times ((3x + z, x^2 - y, xz + yz))$.~~

(c) Explain why it is automatically true that $(3x + z, x^2 - y, xz + yz)$ is not conservative.

(d) Explain why it is automatically true that W is divergence-free, and then verify that this is true by calculating the divergence of W.

Example from Exam Review

(e) Set up an integral that calculates the outward flux of W through the surface $S = \{(x, y, z) : y = 9 - x^2 - z^2 \text{ and } y \geq 0\}$.

(f) Set up a line integral that calculates the same as in the previous question.

(g) Compute the integral in the previous question.

(h) How would this integral be related to the upward flux of W through the disk $\{(x, y, z) : x^2 + z^2 \leq 9, y = 0\}$? Explain the relation and the reason, without calculation.

(12)

$W = z, -z+, x$

14 (a) Is W conservative?

$$\text{curl } W = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ z & (-z+1) & 2x \end{vmatrix} = i\left(\underset{0}{\partial_y(2x)} - \underset{-1}{\partial_z(-z+1)}\right)$$

$$-j\left(\underset{2}{\partial_x(2x)} - \underset{-1}{\partial_z(z)}\right)$$

$$+k\left(\partial_x(-z+1) - \partial_y(z)\right)$$

$\nabla \times W = \text{curl}\, W = (-i - j = (+1, -1, 0)$

(a constant)

Nonzero $\Rightarrow$ W is not conservative.

(b) Show $W = \text{curl}\left(3x+z, x^2-y, xz+yz\right)$

pretend we did it. Yes!

(c) Why must $(3x+z, x^2-y, xz+yz)$ not be conservative? curl $\neq 0$.

(d) Why must $\text{div}\, W = 0$? + Check.

(W is divergence-free).

$$(\text{div}\,(\text{curl}\,(\text{anything}))) = 0$$

$$d(d(\text{form})) = 0$$

$$\text{div}\,(z, -z+1, 2x)$$

$$= (z)_x + (-z+1)_y + (2x)_z$$

$$= 0 + 0 + 0 = \boxed{0}.$$

ⓔ Outward flux of W through surface $S = \{(x,y,z) : y \geq 0, y = 9 - x^2 - z^2\}$

$y = 0$

$0 = 9 - x^2 - z^2$

$x^2 + z^2 = 9$

circle of radius 3 in xz plane.

$W = (z, -z+1, 2x)$

$$\iint_S W \cdot N \, dA$$

$$= \iint_S z \, dy \wedge dz + (-z+1) \, dz \wedge dx + 2x \, dx \wedge dy$$

$y = 9 - x^2 - z^2$

$dy =$ ________ $dx +$ ________ dz

$$\iint_{\substack{\text{circle of} \\ \text{radius } 3}} \left(\qquad \right) dz \wedge dx$$

$$= \int_{r=0}^{3} \int_{\theta=0}^{2\pi} \left(\qquad \right) r \, dr \, d\theta$$